

# The role of biorthogonal partners in sampling theory for non bandlimited signals: a review

Bojan Vrcelj  
Dept. of Electrical Engineering  
California Institute of Technology  
Pasadena, CA 91125  
bojan@systems.caltech.edu

P. P. Vaidyanathan  
Dept. of Electrical Engineering  
California Institute of Technology  
Pasadena, CA 91125  
ppvath@systems.caltech.edu

## Abstract

*It is well known that a broad class of non bandlimited signals can be reconstructed from uniformly spaced samples by relating the problem to the inversion of a digital filter. This simple idea has given rise to many applications, some of them quite sophisticated. This includes spline interpolation, fractionally spaced equalization (FSE) of noisy channels and more recently, rational FSEs. On the theoretical front, it has given rise to the idea of biorthogonal partners which unify many aspects of wavelet theory into the same framework. In this paper we give an overview of the main results in this area.<sup>1</sup>*

## 1 Introduction

The uniform sampling theorem for bandlimited signals is a well-understood concept that is widely used in digital signal processing. However, during the last decade or so it has been noted by many authors [21, 11] that bandlimitedness is not a necessary condition for signal reconstruction from uniform samples. In fact, the bandlimited signal model is just one in the class of signal models that allow for reconstruction. For example, consider a continuous-time signal that can be modeled as

$$x(t) = \sum_{k=-\infty}^{\infty} c(k)\phi(t-k), \quad (1)$$

where  $\phi(t)$  is a known function. If  $\phi(t)$  happens to be bandlimited to some frequency  $\sigma$  then the summation (1) is also bandlimited and  $x(t)$  can be reconstructed from uniform samples taken at the appropriate rate. Now suppose that  $\phi(t)$  is not bandlimited, but satisfies the Nyquist(1) condition; in other words  $\phi(n) = \delta(n)$ . For example  $\phi(t)$  can be the unit-amplitude square pulse supported between  $-1/2$  and  $1/2$ . Then (1) becomes

$$x(t) = \sum_{k=-\infty}^{\infty} x(k)\phi(t-k),$$

which at the same time represents the reconstruction formula for  $x(t)$  from its samples. More generally, as we will

<sup>1</sup>Work supported in part by the ONR grant N00014-99-1-1002, USA.

see in Section 2 the reconstruction of signals admitting the model (1) depends mostly on the properties of the model function  $\phi(t)$  or rather the discrete time Fourier transform of the sampled version  $\phi(n)$ . These extensions of the classical sampling theorem first found application in the all-digital signal interpolation [8]-[10] and least squares signal approximation [9, 15]. Later on they led to the development of theoretical concepts such as *biorthogonal partners* [15], which were further extended to the case of vector signals [19, 20] and fractional biorthogonal partners [16]-[18]. These developments provided the common grounds for some concepts in wavelet theory [2] and signal interpolation, but also shed new light on some equalization methods in digital communications. Examples include the zero-forcing (ZF) channel equalization using fractionally spaced equalizers (FSE) with integer [6] and rational oversampling factors [4, 5]. In this paper we give an overview of the main results in this area with a focus on biorthogonal partners and their applications. Some of the recent overview papers with a special emphasis on the sampling results for non bandlimited signals include [7, 12, 14].

## 1.1 Notations

If not stated otherwise, all notations are as in [13]. Bold faced uppercase letters denote matrices. The term  $\sigma$ -BL refers to signals that are bandlimited to  $|\omega| < \sigma$ . We use the symbol  $\downarrow M$  in a block diagram to denote the decimation operation, which can be applied on scalars or vectors. A decimator turns  $x(n)$  into  $x(Mn)$ . Similarly, the expanded version of  $x(n)$

$$\begin{cases} x(n/M) & \text{for } n = \text{mul of } M, \\ 0 & \text{otherwise} \end{cases}$$

is obtained as a result of the expander operation which is denoted by the symbol  $\uparrow M$ . The decimated and expanded versions of  $x(n)$  are denoted by  $[x(n)]_{\downarrow M}$  and  $[x(n)]_{\uparrow M}$  and the corresponding  $z$ -transforms by  $[X(z)]_{\downarrow M}$  and  $[X(z)]_{\uparrow M}$  respectively. In situations where the  $z$ -transform does not exist in the conventional sense, notation  $z$  stands for  $e^{j\omega}$ . The subscript  $d$  denotes discrete-time signals, whenever there is ambiguity.

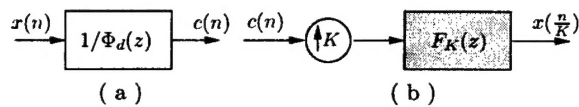


Figure 1: (a)-(b) System for all-digital interpolation of a signal  $x(n)$ .

## 2 Signal models and interpolation

Consider a non bandlimited signal  $x(t)$  and suppose it admits the model (1). Now consider its integer samples

$$x(n) = \sum_{k=-\infty}^{\infty} c(k)\phi(n-k). \quad (2)$$

We see that (2) represents a discrete-time convolution. Denoting the discrete-time Fourier transforms of  $x(n)$ ,  $c(n)$  and  $\phi(n)$  by  $X_d(e^{j\omega})$ ,  $C(e^{j\omega})$  and  $\Phi_d(e^{j\omega})$  respectively, we have  $X_d(e^{j\omega}) = C(e^{j\omega})\Phi_d(e^{j\omega})$ . Therefore, we can retrieve the driving sequence  $c(n)$  from  $x(n)$  by inverse filtering whenever  $\Phi_d(e^{j\omega})$  does not vanish on the unit circle:  $C(e^{j\omega}) = X_d(e^{j\omega})/\Phi_d(e^{j\omega})$ . At the same time, this describes the reconstruction of  $x(t)$  from its integer samples. After obtaining  $c(n)$  by passing  $x(n)$  through  $1/\Phi_d(z)$ , we just need to substitute it in (1) to obtain  $x(t)$ .

From the previous discussion it follows that the class of signals that can be reconstructed from integer samples is much broader than the class of bandlimited signals. As an example, consider any  $\phi(t)$  that is  $\alpha\pi$ -BL, for  $\alpha > 1$ , and such that  $\Phi_d(j\omega)$  is real and positive in the region of support. Then  $\Phi_d(e^{j\omega}) > 0$ , for all  $\omega \in [-\pi, \pi]$  and the reconstruction of  $x(t)$  is possible from samples obtained at the rate  $\alpha$  times lower than the one suggested by the Shannon's result.

Obviously, the successful reconstruction of  $x(t)$  is contingent upon the accuracy of the model (1) and in particular on knowing the model function  $\phi(t)$ . Given an arbitrary signal  $x(n)$  and any function  $\phi(t)$  such that  $\Phi_d(e^{j\omega}) \neq 0$  for all  $\omega$  we can assume that  $x(n)$  is obtained as in (2). On the other hand the assumption that such  $x(n)$  is obtained by sampling  $x(t)$  from (1) is in general not valid for any choice of  $\phi(t)$ . However, in many intrinsically ill-posed problems (such as signal interpolation) there is no "correct" solution, and  $\phi(t)$  is chosen so that the assumed underlying signal  $x(t)$  satisfies some desired properties. Taking signal interpolation as an example, the choice of  $\phi(t)$  is often some smooth function so that the resulting interpolant is visually pleasing. In the case of *spline* interpolation [9, 10]  $\phi(t)$  is taken to be a *B-spline* [3] of a particular order, determining the degree of smoothness, whereas in the case of least squares approximation it is the basis function of the approximation subspace.

In order to understand how interpolation is performed, consider once more the model (1). To interpolate signal  $x(n)$  by an integer factor  $K$  amounts to obtaining the signal  $x(n/K)$  with finer spacing between its samples. Such

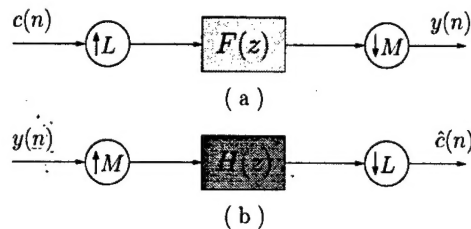


Figure 2: (a)-(b) Interpretation of fractional biorthogonal partners using a block diagram.

signal can be expressed as

$$x(n/K) = \sum_{i=-\infty}^{\infty} c(i)\phi\left(\frac{n-iK}{K}\right) = \sum_{i=-\infty}^{\infty} c(i)f_K(n-iK), \quad (3)$$

where  $f_K(n) \triangleq \phi(n/K)$ . This equality is depicted in Fig. 1(b), while from the previous discussion we recall that the driving sequence  $c(n)$  is obtained from  $x(n)$  as shown in Fig. 1(a), whenever the filtering is stable. It has been shown in [9] that in the case of *B-splines* the filter in question is IIR with half of its poles lying *outside* (but none *on*) the unit circle. Thus  $1/\Phi_d(z)$  can be realized as a two-pass *noncausal* yet stable filter. The beauty of this approach is that the complete interpolation process is performed in the discrete domain. Moreover, once the driving sequence has been obtained, other image processing operations such as rotation, least squares smoothing (denoising) and edge detection can be more easily performed in this domain [10].

## 3 Biorthogonal partners

In the following we review the notion of biorthogonal partners first introduced in [15]. They provide a connection between several different areas of signal processing such as signal interpolation, sampling, least squares modeling and even channel equalization in digital communications. Biorthogonal partners were also introduced in the case of vector signals (MIMO biorthogonal partners) [19]. Apart from that, the idea was extended for the case where signals are oversampled by *fractional* amounts. This gives rise to the notion of fractional biorthogonal partners (FBP) introduced in [17]. Even though biorthogonal partners can be considered as a special case of the latter two, in this section we will interchangeably consider both biorthogonal partners and FBPs in order to point out the important properties pertaining to one or the other.

**Definition 1. Biorthogonal partners.** Two transfer functions  $F(z)$  and  $H(z)$  are said to be biorthogonal partners of each other with respect to an integer  $L$  if

$$P_0(z) \triangleq [H(z)F(z)]_{\downarrow L} = 1. \quad (4)$$

In other words, the convolution of  $h(n)$  and  $f(n)$ , namely  $p(n)$  is Nyquist( $L$ ). It can be shown that  $P_0(z)$  defined in (4) is indeed an LTI system and is nothing but the zeroth

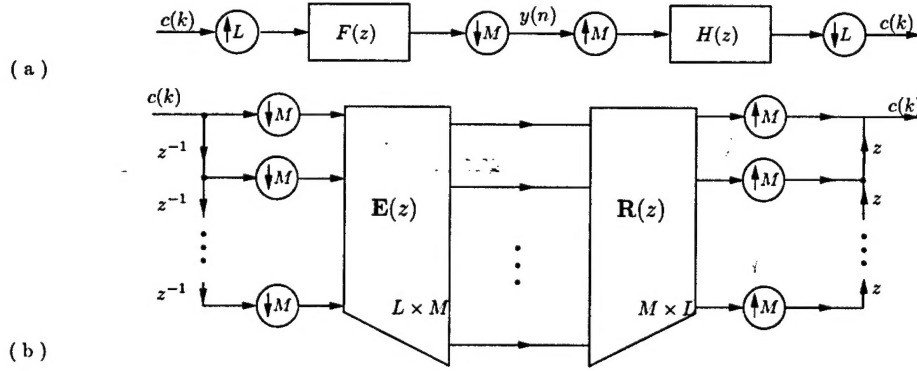


Figure 3: (a)-(b) Equivalent presentations of fractional biorthogonal partners.

$L$ -fold polyphase component [13] of  $P(z)$ . Now consider Fig. 2 with  $L, M > 1$ . Then  $H(z)$  is said to be a *right fractional biorthogonal partner* (RFBP) of  $F(z)$  with respect to the fraction  $L/M$  if  $\hat{c}(n) = c(n)$ , for any  $c(n)$ . In other words, if the system shown in Fig. 3(a) is identity. At the same time we say that  $F(z)$  is a *left fractional biorthogonal partner* (LFBP) of  $H(z)$  with respect to  $L/M$ . Note that whenever  $L$  and  $M$  are coprime<sup>2</sup> the system in Fig. 3(a) is not LTI, so we cannot write its transfer function as in (4). Also note that in the fractional case the biorthogonal partnership is not symmetric, so we distinguish between left and right FBPs.

In order to see how (fractional) biorthogonal partners fit into the sampling framework described before, suppose we are given the discrete-time signal  $y(n)$  that is obtained by sampling  $x(t)$  from (1) at the rate  $L/M$ , i.e.  $y(n) = x(nM/L)$ . Thus  $y(n)$  is obtained by *oversampling*  $x(t)$  by a factor of  $L/M$  with respect to the usual integral sampling strategy. As shown previously, if  $M = L = 1$  (or for that matter if  $M = L$ ) the reconstruction of  $x(t)$  from  $y(n)$  is possible [under some mild conditions on  $\phi(t)$ ], however only by using IIR filters  $1/\Phi_d(z)$ . We shall see shortly that if  $L > M$  the reconstruction is often possible using FIR filters only. First note that

$$y(n) = x\left(\frac{M}{L}n\right) = \sum_{k=-\infty}^{\infty} c(k)f_L(Mn - kL), \quad (5)$$

where  $f_L(t) = \phi(t/L)$ . Thus,  $y(n)$  can be obtained as shown in Fig. 2(a). It is apparent from Definition 1, that the reconstruction of the driving sequence  $c(n)$  [and thus  $x(t)$ ] can be performed as in Fig. 2(b) whenever there exists a RFBP  $H(z)$ . Moreover, if for an FIR  $F(z)$  there exists an FIR  $H(z)$ , the signal reconstruction can be performed using only FIR filters. Next we answer the question when is such (FIR) reconstruction possible. Define the filters  $P_k(z)$  and  $Q_k(z)$  for  $0 \leq k \leq L-1$  as

$$P_k(z) \triangleq z^{kL}F_k(z), \text{ and } Q_k(z) \triangleq z^{-kL}H_k(z). \quad (6)$$

<sup>2</sup>Problem can always be formulated so that  $L$  and  $M$  are coprime by reducing both numbers by their gcd.

Here  $F_k(z)$  and  $H_k(z)$  are the  $L$ -fold Type-2 and Type-1 polyphase components of  $F(z)$  and  $H(z)$ , respectively defined as [13]

$$F(z) = \sum_{k=0}^{L-1} F_k(z^L)z^k, \text{ and } H(z) = \sum_{k=0}^{L-1} H_k(z^L)z^{-k} \quad (7)$$

and  $l$  is an integer such that  $lL + mM = 1$  (for another integer  $m$ ). Recall that such  $m$  and  $l$  exist whenever  $L$  and  $M$  are coprime. Under this assumption it can be shown [17] that the system from Fig. 3(a) is equivalently redrawn in Fig. 3(b). The matrices  $\mathbf{E}(z)$  and  $\mathbf{R}(z)$  consist of the Type-1 and Type-2  $M$ -fold polyphase components of filters  $P_k(z)$  and  $Q_k(z)$ , namely  $E_{i,j}(z)$  and  $R_{i,j}(z)$  defined as

$$P_k(z) = \sum_{j=0}^{M-1} E_{k,j}(z^M)z^{-j}, \text{ and } Q_k(z) = \sum_{i=0}^{M-1} R_{i,k}(z^M)z^i. \quad (8)$$

It follows from Fig. 3 that finding a RFBP of  $F(z)$ , or equivalently reconstructing  $c(n)$  from the samples  $x(n)$ , is completely equivalent to finding a *left inverse* of  $\mathbf{E}(z)$ . The conditions for the existence of an (FIR) solution are summarized in the following theorem [17].

**Theorem 1.** Given the transfer function  $F(z)$  and two coprime integers  $L$  and  $M$ , there exists a *stable* right fractional biorthogonal partner of  $F(z)$  if and only if  $L > M$ , and the minimum rank of  $\mathbf{E}(e^{j\omega})$  pointwise in  $\omega$  is  $M$ . For an FIR filter  $F(z)$  there exists an *FIR* right fractional biorthogonal partner if and only if  $L > M$ , and the greatest common divisor (gcd) of all the  $M \times M$  minors of  $\mathbf{E}(z)$  is a delay. Here, the polyphase matrix  $\mathbf{E}(z)$  is defined by (6)-(8). Analogous results hold for left FBPs as well.

Note that whenever the conditions for the existence of FIR FBPs are satisfied, these solutions are not unique. This is a consequence of the construction for left polynomial inverses of tall polynomial matrices, or equivalently right polynomial inverses of fat polynomial matrices [1]. We will exploit this nonuniqueness in the process of constructing FIR zero-forcing fractionally spaced equalizers for communication channels (see Section 4.1).

In the case of biorthogonal partners ( $M = 1$ ) the inversion problem from Theorem 1 becomes that of finding (FIR) filters  $H_k(z)$ ,  $0 \leq k \leq L - 1$  such that

$$\sum_{k=0}^{L-1} H_k(z) F_k(z) = 1, \quad (9)$$

with  $H_k(z)$  and  $F_k(z)$  denoting the polyphase components defined in (7). It can be shown that in this case a biorthogonal partner of  $F(z)$  exists if and only if  $F_k(z)$  do not share a common zero and an FIR biorthogonal partner exists if and only if the gcd of  $F_k(z)$ ,  $0 \leq k \leq L - 1$  is a delay. In that case, the corresponding FIR filters  $H_k(z)$  can be constructed using the generalization of the Euclid's algorithm. Alternative formulations of the existence conditions for (FIR) biorthogonal partners can be found in [15].

Finally, we note that these results have also been extended in [19, 20] to the case of MIMO biorthogonal partners. For example, in the case of matrix transfer functions, there will exist an FIR LBP of a polynomial matrix  $F(z)$  if and only if the greatest right common divisor (grcd) of the polyphase components  $F_k(z)$  is a unimodular matrix [1]. As was the case with FBPs the construction of FIR MIMO biorthogonal partners (when they exist) is not unique. It has been shown in [20] that finding an FIR LBP is equivalent to finding a left polynomial inverse of a matrix  $[F_0^T(z) \ F_1^T(z) \ \dots \ F_{L-1}^T(z)]^T$ . This problem was further considered in light of the Smith form decomposition [1] and the nonuniqueness of the solution was exploited in the construction of vector channel equalizers. Since the optimization process is very similar to the construction of FIR FBPs, in the next section we will consider only the latter.

## 4 Applications of FBPs

In this section we consider two applications of fractional biorthogonal partners, namely the construction of zero-forcing FSEs with fractional oversampling (a problem that has been considered before in several contexts [4, 5]) and the all-FIR interpolation of slightly oversampled signals.

### 4.1 FSEs with fractional oversampling

It has been shown in [17, 18] that the discrete-time equivalent of the communications system with signal oversampling by a factor of  $L/M$  ( $L > M$ ) at the receiver is essentially given by Fig. 3(a). Here  $F(z)$  is the combined effect of the channel and pulse shaping filters, sampled at rate  $L/T$ , while  $H(z)$  represents the zero-forcing fractionally-spaced equalizer (ZF FSE). In addition to this, the received signal  $y(n)$  is corrupted by additive channel noise. The goal is to construct a ZF FSE [thus a RFBP of  $F(z)$ ] that also attenuates this noise. The construction of such solution is described in [17] and is made possible by the fact that FIR RFBPs are not unique. In the upper part of Fig. 4 we compare the performance of four equalization methods: symbol-spaced equalizer (SSE) which is just the channel inverse, plain RFBP (without noise optimization), optimal RFBP as constructed in [17] and the MMSE solution in the FSE case. Even though its performance is slightly inferior, it turns out that the optimal

RFBP exhibits some computational advantages over the MMSE solution. In particular in the white noise case the optimal RFBP is independent of the estimated noise variance, while the dependency of the MMSE solution on the noise variance discrepancy defined as  $\alpha^2 = \sigma_{est}^2 / \sigma_{act}^2$  is shown in the lower part of Fig. 4.

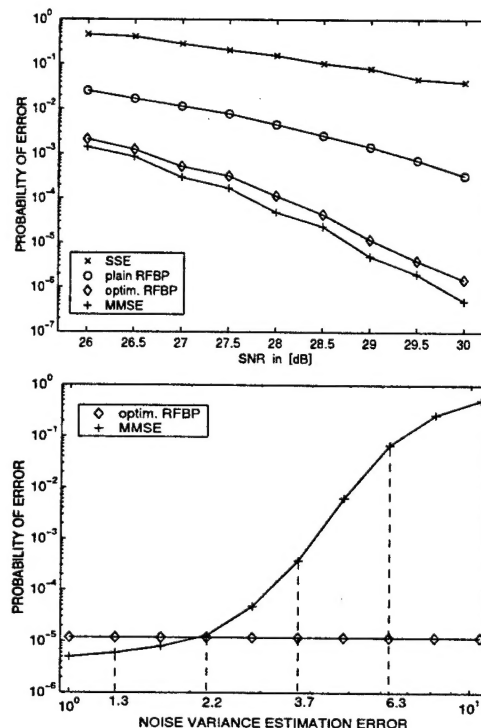


Figure 4: Performance curves. (Upper) Probability of error as a function of SNR in the four equalization methods. (Lower) Probability of error as a function of noise variance discrepancy  $\alpha$ .

### 4.2 Interpolation of oversampled signals

As opposed to traditional spline interpolation described in Section 2 where noncausal IIR filters were used for signal reconstruction, here we show that if the original signal can be assumed to be a spline oversampled by just  $L/M$ , then all-FIR interpolation is possible. To see this, note that if  $y(n) = x(nM/L)$  is a third order spline oversampled by  $L/M$ , then it assumes the model shown in Fig. 2(a), where  $f(n)$  is obtained by sampling the cubic spline at multiples of  $1/L$ . Reconstruction of the driving sequence is possible as shown in Fig. 2(b), where in this case a RFBP  $H(z)$  can be made FIR. Having obtained  $c(n)$  we construct spline interpolants by the total arbitrary amount of  $K \cdot (M/L)$  as shown in Fig. 1(b). An all-FIR interpolation is demonstrated in Fig. 5.

## 5 Concluding remarks

Recent interpretations of sampling theorems for non bandlimited signals have led to several new concepts, like

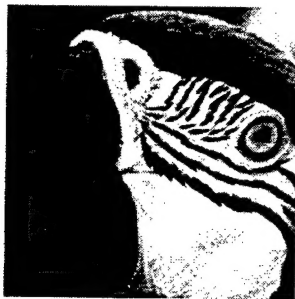


Figure 5: FIR interpolation example: a region of the image oversampled by  $L/M = 6/5$  and its cubic spline interpolation by a factor of  $MK/L = 5/3$  obtained using FIR filters.

biorthogonal partners and fractional biorthogonal partners. Their significance lies in the fact that they help better understand the subtleties of sampling and reconstruction processes and provide a connection between this and several other areas of signal processing. In addition to this, they have given rise to several new applications. In this paper we presented an overview of these issues.

## References

- [1] T. Kailath, *Linear Systems*. Prentice Hall, Inc., Englewood Cliffs, N.J., 1980.
- [2] S. Mallat, *A Wavelet Tour of Signal Processing*, Academic Press, London, 1998.
- [3] I. J. Schoenberg, *Cardinal Spline Interpolation*. SIAM, 1973.
- [4] D. T. M. Slock, "Blind joint equalization of multiple synchronous mobile users using oversampling and/or multiple antennas," in *Proc. 28th Asilomar Conference on Signals, Systems and Computers*, Pacific Grove, CA, Oct. 1994.
- [5] L. Tong, G. Xu and T. Kailath, "A new approach to blind identification and equalization of multipath channels," in *Proc. 25th Asilomar Conf. on Signals, Syst. and Computers*, Pacific Grove, CA, Nov. 1991.
- [6] J. R. Treichler, I. Fijalkow and C. R. Johnson, Jr., "Fractionally spaced equalizers: how long should they be?," *IEEE Signal Proc. Mag.*, pp. 65-81, May 1996.
- [7] M. Unser, "Sampling - 50 years after Shannon," *Proc. of the IEEE*, vol. 88(4), pp. 569-587, April 2000.
- [8] M. Unser, A. Aldroubi, and M. Eden, "Fast B-spline transforms for continuous image representation and interpolation," *IEEE Trans. Patt. Anal. Machine Intell.*, vol. 13, pp. 277-285, Mar. 1991.
- [9] —, "B-spline signal processing: Part I - Theory," *IEEE Trans. Signal Processing*, vol. 41, pp. 821-833, Feb. 1993.
- [10] —, "B-spline signal processing: Part II - Efficient design and applications," *IEEE Trans. Signal Processing*, vol. 41, pp. 834-848, Feb. 1993.
- [11] M. Unser and J. Zerubia, "A generalized sampling theory without bandlimiting constraints," *IEEE Trans. Circ. and Syst. II*, pp. 959-969, Aug. 1998.
- [12] P. P. Vaidyanathan, "Generalizations of the sampling theorem: Seven decades after Nyquist," *IEEE Trans. Circ. and Syst. - I*, vol. 48, pp. 1094-1109, Sep. 2001.
- [13] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*. Prentice-Hall, Englewood Cliffs, NJ, 1995.
- [14] —, "Sampling theorems for non bandlimited signals: theoretical impact and practical applications," *Proc. 4th Intl. Conf. on Sampling Theory and Appl.*, May 2001, Orlando, FL.
- [15] P. P. Vaidyanathan and B. Vrcelj, "Biorthogonal partners and applications," *IEEE Trans. Signal Processing*, vol. 49(5), pp. 1013-1027, May 2001.
- [16] B. Vrcelj and P. P. Vaidyanathan, "Fractional biorthogonal partners and application in signal interpolation," *Proc. ISCAS*, Scottsdale, AZ, May 2002.
- [17] —, "Fractional biorthogonal partners in channel equalization and signal interpolation," *IEEE Trans. Signal Processing*, submitted.
- [18] —, "Fractional biorthogonal partners in fractionally spaced equalizers," *Proc. ICASSP*, Orlando, FL, May 2002.
- [19] —, "MIMO biorthogonal partners and applications," *IEEE Trans. Signal Processing*, vol. 50(3), pp. 528-543, Mar. 2002.
- [20] —, "On the general form of FIR MIMO biorthogonal partners," *Proc. 35th Asilomar Conference on Signals, Systems and Computers*, Pacific Grove, CA, Nov. 2001.
- [21] G. G. Walter, "A sampling theorem for wavelet subspaces," *IEEE Trans. Info. Theory*, vol. 38(2), pp. 881-884, Mar. 1992.